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超函数論におけるエネルギー法

片岡 清臣（東京大学 数理科学研究科）

What is the microlocal energy method?

An L^2 -like method for microfunctions
with C^ω -parameters

Ex. $\begin{cases} \partial f / \partial t = \Delta_x f, & t > 0, x \in \Omega \\ f|_{x \in \partial \Omega} = 0, & t > 0 \end{cases}$

- $f \in B((0, \infty) \times \Omega)$, $\Omega \subset \mathbb{R}^n$; bdd, C^ω -bdry.
- $f|_{x \in \partial \Omega}$ is well-def. $SS(f) \subset \{(t, x; i\tau dt + i\zeta dx) ; \tau > 0\}$
 \downarrow
 x is a C^ω -parameter.

$$\Rightarrow \underline{f \in A((0, \infty) \times \overline{\Omega})}$$

② $\underline{E(t, s) = \int_{\mathbb{R}^n} f(t, x) \overline{f(s, x)} \cdot \chi_\Omega(x) dx} \in B((0, \infty)^2)$
is well-def. because x is a C^ω -parameter for f
up to $x \in \partial \Omega$ (mildness).

Note: Parabolic B.V.P. $\Rightarrow (\partial_t - \partial_s) E(t, s) = 0$

$$\therefore SS(E(t, s)) \not\supset (t^0, t^1; \pm(dt - ds)) \quad (\forall t^0 > 0).$$

\Downarrow Fundamental theorem

$$SS(f(t, x) \chi_\Omega(x)) \not\supset (t^0, x; \pm i\tau dt + i\zeta dx) \\ (\forall t^0 > 0, \forall x, \forall \zeta)$$

Hermitian Positivity for analytic kernels.

S : a set

Def. $K(x, y)$: a \mathbb{C} -valued function on $S \times S$.

K is a hermitian kernel $\Leftrightarrow \overline{K(y, x)} = K(x, y)$ on $S \times S$

" a positive kernel $\Leftrightarrow \forall N, \forall x_1, \dots, x_N \in S$

(Simply we write $K \gg 0$ on $S \times S$) $\left[(K(x_j, x_l))_{j, l} \gg 0 \right]$ (positive semi-def.)

Def. $\Omega \subset \mathbb{C}^n$: open.

$\mathcal{H}(\Omega) := \{ K(z, w) \mid \text{hermitian kernels on } \Omega \times \Omega \text{ \& } K(z, \bar{w}) \in \mathcal{O}(\Omega \times \Omega^*) \}$

analytic h. kernels

\hookrightarrow complex conjugate.

$\mathcal{H}^+(\Omega) := \{ K \in \mathcal{H}(\Omega) \mid K(z, w) \gg 0 \text{ on } \Omega \times \Omega \}$

$\Rightarrow \cdot \mathcal{H}(\Omega)$: \mathbb{R} -algebra (as functions on $\Omega \times \Omega$) with order rel. " \gg ".

$K_1 \gg 0, K_2 \gg 0 \Rightarrow K_1 + K_2, K_1 \cdot K_2 \gg 0$

Facts

• $K_\Omega(z, w)$ (the Bergman kernel) $\in \mathcal{H}^+(\Omega)$

• $\Omega_1 \subset \Omega_2 \Rightarrow 0 \ll K_{\Omega_2}|_{\Omega_1 \times \Omega_1} \ll K_{\Omega_1}$

• $\forall K \in \mathcal{H}(\Omega) \cap L^2(\Omega \times \Omega) \Rightarrow -\|K\|_{L^2} \cdot K_\Omega \ll K \ll \|K\|_{L^2} \cdot K_\Omega$

• (Analytic continuation) $\Omega_1 \subset \Omega_2$: domains

$K_1, K_3 \in \mathcal{H}(\Omega_2), K_2 \in \mathcal{H}(\Omega_1)$ s.t. $K_1 \ll K_2 \ll K_3$

on $\Omega_1 \times \Omega_1$ \Rightarrow $K_2 \in \mathcal{H}(\Omega_2)$ & $K_1 \ll K_2 \ll K_3$ on $\Omega_2 \times \Omega_2$

Fundamental Theorem for microlocal energy methods.

$$\dot{p} = (\dot{x}, \dot{x}; i\dot{\xi} \cdot (dx - du) \in i T^*(\mathbb{R}_x^n \times \mathbb{R}_u^n) \quad (|\dot{\xi}|=1)$$

$$z = x + iy, \quad w = u + iv \text{ (the copy of } z)$$

$$t_j \in T_j(\mathbb{C}U_j) \subset \mathbb{R}^{m_j} : \text{bdd. open, } C^\omega\text{-bdry } (j=1, \dots, N)$$

$$E(x, u) := \sum_{j=1}^N \int_{\mathbb{R}^{m_j}} (Q_j(t_j, x, D_x) + Q_j^*(t_j, u, D_u)) \\ \times (f_j(t_j, x) \cdot \overline{f_j(t_j, u)} \chi_{T_j}(t_j)) p_j(t_j) dt_j$$

$Q_j(t_j, z, \xi)$: holomorphic ψ .D.O. with C^ω -parameter t_j .

s.t. ① Defined on $X_j = \{(t_j, z, \xi) \in \mathbb{C}^{m_j+n+n} \mid t_j \in U_j, \\ |z - \dot{x}| < r, |\frac{z}{|\xi|} - i\dot{\xi}| < r, |\xi| > r^{-1}\}$

② $\exists l_j > \frac{7}{8}, \exists C,$

$$C^{-1} |\xi|^{l_j} \leq \operatorname{Re} Q_j \leq |Q_j| \leq C |\xi|^{l_j} \text{ on } X_j$$

$$Q_j^*(t_j, z, \xi) = \overline{Q_j(\bar{t}_j, \bar{z}, \bar{\xi})}$$

Th. (K; 1985). $p_j(t_j) \in C^\omega(U_j)$, $p_j(t_j) \neq 0$, $p_j(t_j) \geq 0$ ($t_j \in T_j$)
 $f_j(t_j, x) \in B(T_j \times \{|x - \dot{x}| < r\})$ depending C^ω -ly on t_j up to
 $\partial T_j \times \{\dot{x}\}$ (mild sense). Then.

$$\underline{E(x, u) = 0 \text{ at } \dot{p} \Rightarrow \underline{SS(f_j(t_j, x) \chi_{T_j}(u)) \not\equiv (t_j, \dot{x}; i\tau_j dt_j + i\dot{\xi} dx)} \\ (\forall t_j, \forall \tau_j)$$

Positivity for microkernels

$$\dot{p} = (\dot{x}, \dot{z}; i\dot{z} \cdot (dx - dw)) \in \lambda T^*(\mathbb{R}^n \times \mathbb{R}^n)$$

Def $k(x, u) \in C(\mathbb{R}^n \times \mathbb{R}^n / \dot{p})$

a hermitian microkernel $\Leftrightarrow \overline{k(u, x)} = k(x, u)$ at \dot{p}

a positive microkernel $\Leftrightarrow \begin{cases} \exists \Gamma \cap \{y \cdot \dot{z} > 0\} \neq \emptyset \text{ open cone} \\ \exists \underline{k(z, u)} \in \mathcal{H}^+(\{ |z - \dot{x}| < r, \operatorname{Im} z \in \Gamma \}) \\ \text{s.t. } k(x, u) = \underline{k(x + i0\Gamma, u - i0\Gamma)} \text{ at } \dot{p}. \end{cases}$

(Simply we write $k(x, u) \gg 0$ at \dot{p})

Th. " $\gg 0$ at \dot{p} " is an order relation for hermitian microkernels; that is,

$$\underline{k(x, u) \gg 0, -k(x, u) \gg 0 \text{ at } \dot{p} \Leftrightarrow k(x, u) = 0 \text{ at } \dot{p}}$$

$$k_1(x, u) \gg 0, k_2(x, u) \gg 0 \text{ at } \dot{p} \Rightarrow k_1 + k_2 \gg 0 \text{ at } \dot{p}.$$

Ex. • $\delta(x - u) \gg 0$ at $(\dot{x}, \dot{x}; i\dot{z} \cdot (dx - dw)) \forall \dot{x}, \forall \dot{z}$.

$$\int_{\mathbb{R}^m} p(t, x, u, D_x, D_u) (f(t, x) \cdot \overline{f(t, u)} \chi_T(t)) dt \gg 0$$

if " $p(t, x, u, D_x, D_u)$ " is a positive $\psi.d.o.$ with C^ω -parameter t .

Aoki's exp. cal. & ψ .D.O. of restricted hermitian type

Aoki's theory : P, P_1, P_2, P_3 are ψ .D.O. ($\mathcal{E}_{\mathbb{C}^n}^R$) of growth order < 1 .

$$\Rightarrow \left\{ \begin{array}{l} \forall P_1, P_2, \exists P_3 \text{ s.t. } i e^{P_1(z)} : i e^{P_2(z)} = i e^{P_3(z)} \\ \forall P, \exists \tilde{P} (\sim \tilde{P} \equiv P) \text{ s.t. } i e^{P(z)} = e^{i \tilde{P}(z)} \end{array} \right.$$

However, for product h. type ψ .D.O. of order < 1

Aoki's exp. cal. does not work because

$|z|^\sigma \cdot \frac{1}{|y|}$ is not bounded on $V \times V^*$

Def For a product h. type symbol $P(z, w, \bar{z}, \bar{w})$ at \tilde{p} , $i e^{P(z, w, \bar{z}, \bar{w})}$ is said to be of restricted hermitian type at \tilde{p} artificial restr.

$$\Leftrightarrow \left\{ \begin{array}{l} 0 < \underline{\sigma} \leq \frac{1}{2}, \quad \exists C, r > 0 \\ | \text{grad}_{(z, w)} P | \leq C \min \{ |z|^\sigma, |y|^\sigma \} \\ | \text{grad}_{(\bar{z}, \bar{w})} P | \leq C \min \{ |z|^{\sigma-1}, |y|^{\sigma-1} \} \\ \text{on } V_r \times V_r^* \quad \left(\tau \leq C(|z| + |y|)^{\sigma-1} \right) \end{array} \right.$$

In deed, $P = \log(Q_\lambda(z, \bar{z}) + Q_\lambda^*(\bar{w}, \bar{w}))$ O.K.

Examples of P.D.O and Quasi-Positivity

Ex. $P(z, D_z) \in \Sigma_{\mathbb{C}^n}^R | (z; \bar{z})$

$$\Rightarrow \underbrace{P(z, D_z) + P^*(w, D_w)}_{\text{product. h.}}, \underbrace{P(z, D_z) \cdot P^*(w, D_w)}_{\text{positive h.}}$$

Ex. $P(z, D_z)$ as above, & $\exists m, C > 0$.

$$C|z|^m \leq \operatorname{Re} P(z, \bar{z}) \leq |P(z, \bar{z})| \leq C^{-1}|z|^m$$

$$\Rightarrow \underbrace{(P(z, D_z) + P^*(w, D_w))^{-1}}_{\gg 0} \left(= \int_0^\infty e^{-t(P(z, D_z) + P^*(w, D_w))} dt \right) \text{ at } (z, \bar{z}; \bar{z}, z)$$

Here $P(z, D_z) + P^*(w, D_w)$ is not of positive h. type but close to of positive h. type in the following sense :

For a family $P_\lambda + P_\lambda^*$ as above

$$\exists Q(z, w, D_z, D_w) \gg 0, \text{ \& elliptic}$$

$$\text{s.t. } \underbrace{Q \cdot (P_\lambda + P_\lambda^*)}_{\downarrow \text{Quasi-Positivity}} \gg 0 \quad (\forall \lambda) \quad \text{---} \star$$

Write $Q = e^g, \quad P_\lambda + P_\lambda^* = e^{\log(P_\lambda + P_\lambda^*)}$

Then \star is almost equivalent to

$$\underline{g \gg 0, \& \quad g \gg -\log(P_\lambda + P_\lambda^*) \quad (\forall \lambda)}$$

Quasi-positivity of medium restricted h. ψ . D.O.

We add the following conditions to restricted h. ψ . D.O. : $e^{P(\beta, w, \beta, \gamma)}$;

Def Medium $|P(\beta, w, \beta, \gamma)| \leq C(|\beta| + |\gamma|)^{\frac{\sigma}{2}} \text{ on } V \times V^*$

Minimum $|P(\beta, w, \beta, \gamma)| \leq C \min\{|\beta|^\sigma, |\gamma|^\sigma\} \text{ on } V \times V^*$

Indeed, $P = \log(Q_\lambda + Q_\lambda^*)$ falls in $\{\text{Medium}\} \setminus \{\text{Minimum}\}$

(In general, $\{\text{restricted}\} \Rightarrow |P| \leq C(|\beta| + |\gamma|)^\sigma$)

Th (K: 1985)

: $e^{P(\beta, w, \beta, \gamma)}$: one of medium, restricted h. type of order σ ($< \frac{1}{2}$) with a common constant $C > 0$

$\Rightarrow \sigma < \forall \sigma' < \frac{1}{2}$, $\exists p \gg 0$ & e^p is of minimum & restricted h. type of order σ'

s.t. $|e^p| : |e^{P_\lambda}| \gg 0 \ (\forall \lambda)$

Ex. $0 < \sigma < \frac{1}{2}$, $N = 2, 3, 4, \dots$

$P = \frac{(\beta\gamma)^{N+2}}{(\beta^N + \gamma^N)^2} \gg 0$ & e^p is of minimum & restricted h. type

Our Conjecture

We want to remove the order restriction $(0 < \sigma < \frac{1}{2})$ in the preceding theorem. To do so, we extend the conditions on restricted h.type:

Def (J. Kumagai, 2000: Master thesis)

For a product h. type symbol $P(z, w, \xi, \eta)$ at $p^0 : e^{P(z, w, \xi, \eta)}$: is said to be of m-restricted h. type ($m = 1, 2, \dots$)

$$\Leftrightarrow \left\{ \begin{array}{l} 0 < \exists \sigma < \frac{m}{m+1}, \exists C, \exists r > 0 \\ | \text{grad}_{(z, w)} P | \leq C \min \{ |\xi|^r, |\eta|^r \} \\ | \partial_z^\alpha \partial_\eta^\beta P | \leq C \min \{ |\xi|^{\sigma - |\alpha| - |\beta|}, |\eta|^{\sigma - |\alpha| - |\beta|} \} \\ \quad (\forall \alpha, \beta \text{ s.t. } 0 < |\alpha| + |\beta| \leq m) \text{ on } V \times V^* \end{array} \right.$$

Conjecture : $e^{P_2(z, w, \xi, \eta)}$ of medium, m-rest. h. type of order $\sigma (< \frac{m}{m+1})$

$$\Rightarrow \sigma < \forall \sigma' < \frac{m}{m+1}, \exists p \gg 0 \text{ \& } e^p \text{ is of minimum \& m-restricted, order } \sigma' \text{ s.t. } |e^p| : |e^{P_\lambda}| \gg 0 \quad (\forall \lambda)$$

C. H. Lee's results

He generalized minimum type Ψ .D.O. to any product space $X \times Y$ and proved that Aoki calculus works well for such $\exists e^{P(\mathbb{R}, w, \mathbb{Z}, \gamma)}$: i.e.

$$|P(\mathbb{R}, w, \mathbb{Z}, \gamma)| \leq C \min \{ \Lambda_1(|\mathbb{Z}|), \Lambda_2(|\gamma|) \}$$

where $\Lambda_1(t), \Lambda_2(t)$ are infra-linear weight functions.

Th (C. H. Lee, 2003)

For minimum type Ψ .D.O. on $X \times Y$, we have

$$\textcircled{1} \quad \forall e^{P_1(\mathbb{R}, w, \mathbb{Z}, \gamma)} \quad \forall e^{P_2(\mathbb{R}, w, \mathbb{Z}, \gamma)} \quad = \quad \exists e^{P_3(\mathbb{R}, w, \mathbb{Z}, \gamma)}$$

$$\textcircled{2} \quad \forall e^{P(\mathbb{R}, w, \mathbb{Z}, \gamma)} \quad = \quad \exists \tilde{P}(\mathbb{R}, w, \mathbb{Z}, \gamma)$$

Conversely

$$\exists e^{P(\mathbb{R}, w, \mathbb{Z}, \gamma)} \quad = \quad \forall e^{\tilde{P}(\mathbb{R}, w, \mathbb{Z}, \gamma)}$$

Proofs are written in formal symbols.

Lee's Proof (notation change $\boxed{z, w, z, \eta \Rightarrow z, z^*, \xi, \xi^*}$)

$$\textcircled{1} : e^{p(t; z, \xi)} \dots e^{g(t; z, \xi)};$$

$$= \exp(t_2 \partial_z \cdot \partial_w + t_2^* t^* \partial_{z^*} \cdot \partial_{w^*}) \left(\exp(p(t; z, \xi) + g(t; w, \eta)) \right)$$

\parallel
 G

$t_2 = t_2^* = 1 \quad w = z, \eta = \xi$
 $w^* = z^*, \eta^* = \xi^*$

Hence, it is sufficient to calculate G .

Further, express G as

$$G = \exp\left(\sum_{k, k^*} t_2^k t_2^{*k^*} r_{k, k^*}(t; z, \xi, w, \eta; t^*; z^*, \xi^*, w^*, \eta^*)\right)$$

Then we have the following recurrence formulas

$$\begin{cases} r_{0,0} = p(t; z, \xi) + g(t; w, \eta) \\ r_{k+1, k^*} = \frac{t}{k+1} \left\{ \partial_z \cdot \partial_w r_{k, k^*} + \sum_{\substack{k'+k''=k \\ k'^*+k''^*=k^*}} \partial_z r_{k', k'^*} \cdot \partial_w r_{k'', k''^*} \right\} \\ r_{k, k^*+1} = \frac{t^*}{k^*+1} \left\{ \text{similar terms} \right\} \end{cases}$$

The difference from Aoki's estimate:

$$r_{k, k^*} = \sum_{j=k, j^*=k^*}^{\infty} t^j t^{j^*} \gamma_{(j, k), (j^*, k^*)}$$

NO USE of l, l^* in the formulas

That is, we have the following estimates:

$$|r_{(j, k), (j^*, k^*)}| \leq \sum_{\substack{0 \leq l \leq k \\ 0 \leq l^* \leq k^*}} B^{k+k^*} A^{j-k+l+j^*-l^*} (j+1)^{-2} (j^*+1)^{-2} \\ \times (k+1)^{k-l-3} (k^*+1)^{k^*-l^*-3} \varepsilon^{-2k} \varepsilon^{*-2k^*} |\xi|^{-k} |\xi^*|^{-k^*} \\ \times (\Lambda(|\xi|) + \Lambda(|\eta|))^l \cdot (\Lambda^*(|\xi^*|) + \Lambda^*(|\eta^*|))^{l^*} (\tilde{\Lambda}(\xi, \xi^*) + \tilde{\Lambda}(\eta, \eta^*))$$

Here $\tilde{\Lambda}(\xi, \xi^*) = \min \{ \Lambda(|\xi|), \Lambda^*(|\xi^*|) \}$

We need l, l^* , but " $r_{(j, k), (j^*, k^*)}^{l, l^*}$ " does not work. Further, In the estimates of non-linear terms, Replace one of $(\tilde{\Lambda}(\xi, \xi^*) + \tilde{\Lambda}(\eta, \eta^*))^2$ factor
 by $\Lambda(|\xi|) + \Lambda(|\eta|)$ or $\Lambda^*(|\xi^*|) + \Lambda^*(|\eta^*|)$